Solutions to HW #11

1. $\nabla f(x,y) = \left(\frac{3}{2}xy^4 - \frac{5}{4}y^6,\ 3x^2y^3 - \frac{15}{2}xy^5\right)$

2. $\nabla g(x,y) = \left(\sin y - 2\cos x \sin x\, y,\ x\cos y + \cos^2 x\right) =$

$= \left(\sin y - y\sin 2x,\ x\cos y + \cos^2 x\right)$

3. $\nabla h(x,y,z) = \left(e^{yz},\ xze^{yz},\ xye^{yz}\right)$

4. $\nabla H(x_1,\dots,x_n) = \left(\frac{2x_1}{\sum_{i=1}^{n} x_i^2},\ \dots,\ \frac{2x_n}{\sum_{i=1}^{n} x_i^2}\right)$

5. $\nabla P(x_1,\dots,x_n) = \left(2x_1 e^{\sum_{i=1}^{n} x_i^2},\ \dots,\ 2x_n e^{\sum_{i=1}^{n} x_i^2}\right)$

6. $\nabla f(x_1,\dots,x_n) = \left(2x_1,\dots,2x_n\right)$

7. The unit directional vector is $u = \frac{v}{\|v\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

$\nabla f(x,y) = \left(2\cdot 7x + 4y,\ 4x - 2\cdot 3y\right)$

$\nabla f\left(\frac{2}{3}, \frac{3}{2}\right) = \left(\frac{46}{3}, \frac{-19}{3}\right)$.

$D_u f\left(\frac{2}{3}, \frac{3}{2}\right) = \nabla f\left(\frac{2}{3}, \frac{3}{2}\right)\cdot\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \left(\frac{46}{3}, -\frac{19}{3}\right)\cdot\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) =$

$= \frac{46}{3\sqrt{5}} - \frac{38}{3\sqrt{5}} = \frac{8}{3\sqrt{5}}$.

8. The unit directional vector is $u = \frac{v}{\|v\|} = \left(\frac{-3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\right)$

$\nabla f(x,y) = \left(\frac{2}{3}x^{-\frac{1}{3}}, \frac{2}{3}y^{-\frac{1}{3}}\right)$; $\nabla f(1,2) = \frac{2}{3}\left(1, \frac{1}{\sqrt[3]{2}}\right)$

$D_u f(1,2) = \frac{2}{3}\left(1, \frac{1}{\sqrt[3]{2}}\right)\cdot\frac{1}{\sqrt{34}}\left(-3, 5\right) = \frac{2}{3\sqrt{34}}\left(-3 + \frac{5}{\sqrt[3]{2}}\right)$

9. $u = \frac{v}{\|v\|} = \frac{1}{\sqrt{e^2+1}}(e,1)$

$\nabla f(x,y) = \left(\frac{1}{x+e^y}, \frac{e^y}{x+e^y}\right)$; $\nabla f(e,1) = \left(\frac{1}{2e}, \frac{1}{2}\right)$

$D_u f(e,1) = \left(\frac{1}{2e}, \frac{1}{2}\right) \cdot \frac{1}{\sqrt{e^2+1}}(e,1) = \frac{1}{\sqrt{e^2+1}}$

10. $v = (0,0,0) - (3,-4,1) = (-3,4,1)$.

$u = \frac{v}{\|v\|} = \frac{1}{\sqrt{26}}(-3,4,1)$

$\nabla f(x,y,z) = \left(\frac{x}{z\sqrt{x^2+y^2}}, \frac{y}{z\sqrt{x^2+y^2}}, \frac{-\sqrt{x^2+y^2}}{z^2}\right)$

$\nabla f(-3,4,1) = \left(\frac{-3}{5}, \frac{4}{5}, -5\right)$

$D_u f(-3,4,1) = \nabla f(-3,4,1) \cdot \frac{1}{\sqrt{26}}(-3,4,1) = \left(\frac{-3}{5}, \frac{4}{5}, -5\right) \cdot$

$\cdot \frac{1}{\sqrt{26}}(-3,4,1) = 0.$

11. $v = (3,6,e) - (2,e,1/e) = (1, 6-e, e-1/e)$

Thus $u = \frac{(1, 6-e, e-1/e)}{\sqrt{1+(6-e)^2+(e-1/e)^2}}$.

$\nabla f(x,y,z) = \left(\frac{yz}{xyz+1}, \frac{xz}{xyz+1}, \frac{xy}{xyz+1}\right)$; $\nabla f(2,e,1/e) = \left(\frac{1}{3}, \frac{2/e}{3}, \frac{2e}{3}\right)$

$= \frac{1}{3}\left(1, \frac{2}{e}, 2e\right)$

$D_u f(2,e,1/e) = \frac{1}{3\sqrt{1+(6-e)^2+(e-1/e)^2}}\left(\frac{12-2e}{e} + 2e^2 - 1\right)$

12. a) $\nabla T(x,y,z) = (-2x, -4y, -6z)\, e^{-x^2-2y^2-3z^2}$

$\nabla T(1,1,1) = (-2,-4,-6)\, e^{-6}$

To decrease the temperature most rapidly, the ship must travel in the opposite direction to the gradient:

$$u = \frac{-\nabla T(1,1,1)}{\|\nabla T(1,1,1)\|} = \frac{1}{\sqrt{14}}(1,2,3)$$

b) If the ship travels in the direction computed in (a), the rate of change in temperature per meter is $\frac{-2e^{-6}\sqrt{14}}{1}\left(C^\circ/m\right)$

To compute the cooling rate with respect to time, observe that
$C^\circ/t = C^\circ/m \cdot m/t. = \frac{-2e^{-6}\sqrt{14}}{1} \cdot \frac{1}{e^{-8}} = -2e^2\sqrt{14}\ \left(C^\circ/sec\right)$

Thus, the maximal rate of cooling is $2e^2\sqrt{14}$ degrees per second.

c) To cool as rapidly as the ship structure would allow, the ship must head at an angle θ to the gradient. This angle must satisfy the equation

$$\|\nabla T(1,1,1)\| \cos\theta = -e^2\sqrt{14}$$

Thus

$$2e^2\sqrt{14}\cos\theta = -e^2\sqrt{14}$$

$$\cos\theta = \frac{-1}{2}$$

It follows that the ship should move in any direction that makes an angle of $\frac{2\pi}{3}$ radians with the gradient. The set of possible solutions is therefore a cone.

13. $\nabla T(x,y) = (4x, -8y) = 4(x,-2y)$.

$\nabla T(-1,2) = 4(-1,-4)$.

The bug should move in the direction $\dfrac{-\nabla T(-1,2)}{\|\nabla T(-1,2)\|} =$

$= \dfrac{(1,4)}{\sqrt{17}}$.

14. $\nabla Z(x,y) = (-2ax, -2by) = -2(ax, by)$

$\nabla Z(1,1) = -2(a,b)$.

Let θ be an angle measured from the gradient to the directional vector, along which the track is to be laid. Then θ must satisfy the equation

$$\|\nabla Z(1,1)\| \cos\theta = 0.03$$

$$2\sqrt{a^2+b^2}\cos\theta = 0.03$$

$$\theta_1 = \cos^{-1}\left(\frac{0.03}{2\sqrt{a^2+b^2}}\right) \text{ or } \theta_2 = -\cos^{-1}\left(\frac{0.03}{2\sqrt{a^2+b^2}}\right)$$

The directional derivatives are therefore

$$u_1 = (\cos(\varphi+\theta_1), \sin(\varphi+\theta_2)) \text{ and } u_2 = (\cos(\varphi+\theta_2), \sin(\varphi+\theta_2))$$

where φ is the angle that the gradient makes with respect to the x-axis (How can φ be computed?)

15. Let $F(x,y,z) = x^2+y^2-z^2$. Then, for any point (x,y,z) on the surface $x^2+y^2-z^2=-1$, $\nabla F(x,y,z) = 2(x,y,-z)$ is normal to the surface. Since any surface is two-dimensional, all possible normal vectors to the surface are scaler multiples of $\nabla F(x,y,z)$.

Thus, the two alternatives for the direction of the partical motion are $\frac{\nabla f(1,1,\sqrt{3})}{\|\nabla f(1,1,\sqrt{3})\|} = \frac{(1,1,-\sqrt{3})}{\sqrt{5}}$ or $\frac{-\nabla f(1,1,\sqrt{3})}{\|\nabla f(1,1,\sqrt{3})\|} =$

$= \frac{(-1,-1,\sqrt{3})}{\sqrt{5}}$. Since the particle must "fall" on the xy plane, the first alternative is the desired direction. Since the speed of the particle is 10 units per second, the path of the particle may be described by $l(t) = (1,1,\sqrt{3}) + \frac{10t}{\sqrt{5}}(1,1,-\sqrt{3})$.

When $\sqrt{3} - \sqrt{3}\left(\frac{10t}{\sqrt{5}}\right) = 0$, the particle will have collided with the xy plane. This happens when $\frac{10}{\sqrt{5}}t = 1$. Thus $t = \frac{\sqrt{5}}{10}$.

At this time $l\left(\frac{\sqrt{5}}{10}\right) = (1,1,\sqrt{3}) + (1,1,-\sqrt{3}) = (2,2,0)$.

16. a) $\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

b) Because $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, f is differentiable at $(0,0)$ iff $Df(0,0)(x,y) = 0$

But $\lim_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-0|}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{|hk||k|}{\sqrt{h^2+k^2}\,(h^2+k^2)} =$

$= \lim_{r\to 0^+} \frac{\frac{1}{2}r^3|\sin 2\theta \sin\theta|}{r^3} = \frac{1}{2}|\sin 2\theta \sin\theta|$.

Since the limit depends on θ, it does not exist.

Thus, f cannot be differentiable at $(0,0)$.

c) We have already seen that the directional derivatives along the x and y axes exist and equal 0.

Let $u = (a, b)$, $a, b \neq 0$ then

$$D_u f(0,0) = \lim_{t \to 0} \frac{f((0,0)+t(a,b)) - f(0,0)}{t} = \lim_{t \to 0} \frac{\frac{(at)(bt)^2}{(at)^2+(bt)^2}}{t} =$$

$$= \lim_{t \to 0} \frac{t^3 ab}{t^3(a^2+b^2)} = \frac{ab}{a^2+b^2} \neq 0$$

In particular, all directional derivatives exist.

Notice however, that $D_u f(0,0) \neq \nabla f(0,0) \cdot u = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)\right) \cdot (a, b) = (0,0) \cdot (a,b) = 0.$